RateMeasure Approximations & Increasing-Decreasing Functions

1 Marks Questions

1. The amount of pollution content added in air in a city due to x diesel vehicles is given by

$$P(x) = 0.005x^3 + 0.02x^2 + 30x.$$

Find the marginal increase in pollution content when 3 diesel vehicles are added and write which value is indicated in the above question? Delhi 2013; Value Based Question

Given, $P(x) = 0.005x^3 + 0.02x^2 + 30x$

On differentiating both sides w.r.t. x, we get

$$P'(x) = 3 \times 0.005x^2 + 2(0.02)x + 30$$

On putting $x = 3$, we get

$$P'(3) = 3 \times 0.005 \times 9 + 2(0.02)(3) + 30$$

$$= 0.135 + 0.12 + 30 = 30.255 \ \textbf{(1/2)}$$

Pollution content in the city increases with the increase in number of diesel vehicles. **(1/2)**

2. The total cost $C(x)$ associated with provision of free mid-day meals to x students of a school in primary classes is given by

$C(x) = 0.005x^3 - 0.02x^2 + 30x + 50.$

If the marginal cost is given by rate of change $\frac{dC}{dx}$ of total cost, then write the marginal cost of food for 300 students. What value is shown here?

Delhi 2013C; Value Based Question

Given, $C(x) = 0.005x^3 - 0.02x^2 + 30x + 50$

On differentiating both sides w.r.t. x, we get

$$\frac{dC}{dx} = 0.005(3x^2) - 0.02(2x) + 30$$

$$= 0.015x^2 - 0.04x + 30$$

On putting $x = 300$, we get

$$\frac{dC}{dx} = 0.015(300)^2 - 0.04(300) + 30$$

$$= 1350 - 12 + 30 = 1368$$ **(1/2)**

By providing free mid-day meals to students of primary classes, care and concern is shown towards their health and nutritional status. **(1/2)**

3. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue).

If the total revenue (in ₹) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, then find the marginal revenue, when $x = 5$ and write which value does the question indicate.

All India 2012; Value Based Question

We know that, marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$$\therefore \text{Marginal revenue (MR)} = \frac{dR}{dx}$$

$$= \frac{d}{dx}(3x^2 + 36x + 5) = 6x + 36$$

When $x = 5$, then

$$MR = 6(5) + 36 = 30 + 36 = 66$$

Hence, the required marginal revenue is ₹ 66.

(1/2)

More amount of money spent for the welfare of the employees with the increase in marginal revenue. **(1/2)**

4 Marks Questions

4. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is

(i) strictly increasing.

(ii) strictly decreasing. Delhi 2014

Given function is

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

On differentiating w.r.t. x, we get

$$f'(x) = 12x^3 - 12x^2 - 24x \quad \textbf{(1)}$$

On putting $f'(x) = 0$, we get

$$12x^3 - 12x^2 - 24x = 0$$

$$\Rightarrow \quad 12x(x^2 - x - 2) = 0$$

$$\Rightarrow \quad 12\,x\,[x^2 - 2x + x - 2] = 0$$

$$\Rightarrow \quad 12\,x\,(x + 1)\,(x - 2) = 0$$

$$\Rightarrow \quad x = 0, -1 \text{ or } 2 \quad \textbf{(1)}$$

Now, we find intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$f'(x) = 12x(x+1)(x-2)$	Sign of $f'(x)$
$x < -1$	(–) (–) (–)	– ve
$-1 < x < 0$	(–) (+) (–)	+ ve
$0 < x < 2$	(+) (+) (–)	– ve
$x > 2$	(+) (+) (+)	+ ve

We know that, a function $f(x)$ is said to be strictly increasing, if $f'(x) > 0$ and it is said to be strictly decreasing, if $f'(x) < 0$. So, the given function $f(x)$ is

(i) strictly increasing on the intervals $[-1, 0]$ and $[2, \infty)$.

(ii) strictly decreasing on the intervals $(-\infty, -1]$ and $[0, 2]$. **(2)**

5. Find the intervals in which the function given by $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36x}{5} + 11$ is

(i) strictly increasing.

(ii) strictly decreasing   All India 2014C

Given function is

$$f(x) = \frac{3x^4}{10} - \frac{4x^3}{5} - 3x^2 + \frac{36x}{5} + 11$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \frac{12x^3}{10} - \frac{12x^2}{5} - 6x + \frac{36}{5} + 0 \quad \textbf{(1)}$$

On putting $f'(x) = 0$, we get

$$\frac{12x^3}{10} - \frac{12x^2}{5} - 6x + \frac{36}{5} = 0$$

$$\Rightarrow \quad \frac{6}{5} \cdot (x-1)(x+2)(x-3) = 0$$

$$\Rightarrow \quad x - 1 = 0 \text{ or } x + 2 = 0$$

$$\text{or} \quad x - 3 = 0$$

$$\Rightarrow \quad x = -2, 1, 3 \quad \textbf{(1)}$$

Now, we find intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$f'(x) = \frac{6}{5}(x-1)(x+2)(x-3)$	Sign of $f'(x)$
$(-\infty, 2]$	$(-)(-)(-)$	$-$ ve
$[-2, 1]$	$(-)(+)(-)$	$+$ ve
$[1\ 3]$	$(+)(+)(-)$	$-$ ve
$[3, \infty)$	$(+)(+)(+)$	$+$ ve

We know that, a function $f(x)$ is said to be strictly increasing, if $f'(x) > 0$ and decreasing, if $f'(x) < 0$. So, the given function $f(x)$ is

(i) strictly increasing in $[-2, 1] \cup [3, \infty)$.

(ii) strictly decreasing in $(-\infty, -2] \cup [1, 3]$. **(2)**

6. Find the value(s) of x for which $y = [x(x-2)^2]$ is an increasing function.

All India 2014

Given function is $y = [x(x-2)]^2 = [x^2 - 2x]^2$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2(x^2 - 2x)\frac{d}{dx}(x^2 - 2x)$$

$$= 2(x^2 - 2x)(2x - 2) = 4x(x-2)(x-1) \quad \textbf{(1)}$$

On putting $\frac{dy}{dx} = 0$, we get

$4x(x-2)(x-1) = 0$

$\Rightarrow \qquad x = 0, 1 \text{ and } 2$ **(1)**

Now, we find interval in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	Sign of $f(x)$	Nature of $y(x)$
$(-\infty, 0]$	$(-)(-)(-) = -$ ve	Strictly decreasing
$[0, 1]$	$(+)(-)(-) = +$ ve	Strictly increasing
$[1, 2]$	$(+)(-)(+) = -$ ve	Strictly decreasing
$[2, \infty)$	$(+)(+)(+) = +$ ve	Strictly increasing

Hence, y is increasing in $[0, 1]$ and $[2, \infty)$, i.e. $x \in (0, 1)$ and $(2, \infty)$. **(2)**

7. Using differentials, find the approximate value of $(3.968)^{3/2}$.

Delhi 2014C

Let $y = f(x) = (x)^{3/2}$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \frac{3}{2} \cdot x^{1/2} \quad \textbf{(1)}$$

Let $\qquad x = 4 \quad$ and $\quad x + \Delta x = 3.968$

Then, $\qquad \Delta x = -0.032$ **(1)**

Now, $f(x + \Delta x) = f(x) + f'(x)\Delta x$

$$\therefore \quad (x + \Delta x)^{3/2} = (x)^{3/2} + \frac{3}{2} \cdot (x)^{1/2} \cdot (-0.032) \quad \textbf{(1)}$$

$$\Rightarrow (4 - 0.032)^{3/2} = (4)^{3/2} + \frac{3}{2} \cdot (4)^{1/2} \cdot (-0.032)$$

$$\Rightarrow \quad (3.968)^{3/2} = 8 + \frac{3}{2} \cdot 2 \cdot (-0.032)$$

$$= 8 - 0.096 = 7.904 \quad \textbf{(1)}$$

8. Find the intervals in which the function

$$f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51 \text{ is}$$

(i) strictly increasing.

(ii) strictly decreasing. Foreign 2014C

Given function is

$$f(x)=\frac{3}{2}x^4-4x^3-45x^2+51$$

On differentiating w.r.t. x, we get

$$f'(x)=6x^3-12x^2-90x$$

$$=6x(x^2-2x-15) \qquad \textbf{(1)}$$

Now, on putting $f'(x)=0$, we get

$$6x(x^2-2x-15)=0$$

$$\Rightarrow \quad 6x(x^2-5x+3x-15)=0$$

$$\Rightarrow \quad 6x(x-5)(x+3)=0$$

$$\Rightarrow \quad x=-3,0,5 \qquad \textbf{(1)}$$

Now, we find intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$f'(x)=6x(x-5)(x+3)$	**Sign of $f'(x)$**
$(-\infty,-3]$	(–) (–) (–)	– ve
$[-3, 0]$	(–) (–) (+)	+ ve
$[0, 5]$	(+) (–) (+)	– ve
$[5, \infty)$	(+) (+) (+)	+ ve

(1)

We know that, a function $f(x)$ is said to be strictly increasing, if $f'(x)>0$ and decreasing, if $f'(x)<0$. So, the given function $f(x)$ is

(i) strictly increasing in $[-3,0]\cup[5,\infty)$.

(ii) strictly decreasing in $(-\infty,-3]\cup[0,5)$. **(1)**

9. Find the approximate value of $f(3.02)$, upto 2 places of decimal, where $f(x)=3x^2+5x+3$.

Foreign 2014

Firstly, split 3.02 into two parts x and Δx, so that $x+\Delta x=3.02$ and $f(x+\Delta x)=f(3.02)$.
Now, write $f(x+\Delta x)\approx f(x)+\Delta x\cdot f'(x)$ and use this result to find the required value.

Given function is $f(x) = 3x^2 + 5x + 3$

On differentiating w.r.t.x, we get $f'(x) = 6x + 5$

Let $x = 3$ and $\Delta x = 0.02$ **(1)**

So that $f(x + \Delta x) = f(3.02)$

By using $f(x + \Delta x) \approx f(x) + \Delta x\, f'(x)$, we get

$f(3.02) = 3x^2 + 15x + 3 + (6x + 15) \cdot \Delta x$ **(1)**

$= 3(3)^2 + 15(3) + 3 + [6(3) + 15](0.02)$ **(1)**

$= 27 + 45 + 3 + 33(0.02)$

$= 75 + 0.66 = 75.66$

Hence, $f(3.02) = 75.66$ **(1)**

10. Using differentials, find approximate value of $\sqrt{49.5}$. **Delhi 2012**

Firstly, divide 49.5 into two parts as $x = 49$ and $\Delta x = 0.5$. Now, let $y = \sqrt{x}$ and find $\frac{dy}{dx}$. Finally, find Δy using the formula $dy = \frac{dy}{dx} \cdot \Delta x$.

Let $x = 49$, $\Delta x = 0.5$ and $y = \sqrt{x}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \quad \textbf{(1)}$$

At $x = 49$, $\left[\frac{dy}{dx}\right]_{x=49} = \frac{1}{2\sqrt{49}} = \frac{1}{2 \times 7} = \frac{1}{14}$ **(1)**

We know that, $dy = \frac{dy}{dx} \cdot \Delta x$

$$\Rightarrow \quad dy = \frac{1}{14} \times 0.5 = \frac{5}{140} = \frac{1}{28} \quad \textbf{(1)}$$

$$\therefore \quad \sqrt{49.5} \approx y + \Delta y = \sqrt{49} + \frac{1}{28} = 7 + \frac{1}{28}$$

$$[\because y = \sqrt{x} = \sqrt{49} = 7]$$

$$= \frac{196 + 1}{28} = \frac{197}{28} = 7.035 \quad \textbf{(1)}$$

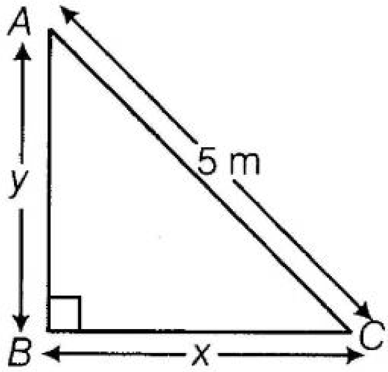
Hence, approximate value of $\sqrt{49.5}$ is 7.035.

11. A ladder 5 m long is leaning against a wall. Bottom of ladder is pulled along the ground away from wall at the rate of 2 m/s. How fast is the height on the wall decreasing, when the foot of ladder is 4 m away from the wall?

HOTS; All India 2012

The ladder when leaning against the wall forms a right angled triangle with length of ladder equal to the hypotenuse of the triangle and base and altitude equal to the distance of foot from wall and distance of top from ground. Further, use the Pythagoras theorem to make an equation and then differentiate it.

Let AC be the ladder, $BC = x$ and height of the wall, $AB = y$.



As the ladder is pulled along the ground away from the wall at the rate of 2 m/s. So,

$$\frac{dx}{dt} = 2 \text{ m/s}$$

To find $\frac{dy}{dt}$, when $x = 4$. **(1)**

In right-angled ΔABC, by Pythagoras theorem, we get

$$(AB)^2 + (BC)^2 = (AC)^2$$

$$\Rightarrow \quad x^2 + y^2 = 25 \quad \ldots\text{(i)}$$

$$\Rightarrow \quad (4)^2 + y^2 = 25$$

$$\Rightarrow \quad 16 + y^2 = 25 \Rightarrow y^2 = 9$$

$$\Rightarrow \quad y = \sqrt{9}$$

$$\therefore \quad y = 3$$ **(1)**

On differentiating both sides of Eq. (i) w.r.t. t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

[dividing both sides by 2]

On substituting the values of x, y and $\frac{dx}{dt}$, we get

$$(4 \times 2) + 3 \times \frac{dy}{dt} = 0$$

$$\Rightarrow \quad 8 + 3 \times \frac{dy}{dt} = 0$$ **(1)**

$$\frac{dy}{dt} = \frac{-8}{3} \text{ m/s}$$

Hence, height of the wall is decreasing at the rate of $\frac{8}{3}$ m/s. **(1)**

NOTE In a rate of change of a quantity, +ve sign shows that it is increasing and –ve sign shows that it is decreasing.

12. Show that $y = \log(1+x) - \frac{2x}{2+x}, x > -1$ is an increasing function of x, throughout its domain. Foreign 2012

Given function is $y = \log(1+x) - \frac{2x}{2+x}$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+x}(1) - \frac{(2+x)\cdot 2 - 2x \cdot 1}{(2+x)^2} \quad \textbf{(1)}$$

$$= \frac{1}{1+x} - \frac{4+2x-2x}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{4+x^2+4x-4-4x}{(1+x)(2+x)^2}$$

$$= \frac{x^2}{(1+x)(2+x)^2} \quad \text{...(i)}$$

(1½)

Now, $x^2, (2+x)^2$ are always positive, also $1+x > 0$ for $x > -1$. **(1/2)**

From Eq. (i), $\frac{dy}{dx} > 0$ for $x > -1$.

Hence, function increases for $x > -1$. **(1)**

13. Find the intervals in which the function given by $f(x) = \sin x + \cos x, 0 \le x \le 2\pi$ is

(i) increasing. (ii) decreasing. Delhi 2012C

Given function is $f(x) = \sin x + \cos x$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x - \sin x \quad \textbf{(1)}$$

Now, put $f'(x) = 0$

$\Rightarrow \cos x - \sin x = 0 \Rightarrow \tan x = 1$

$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$, as $0 \le x \le 2\pi$ **(1)**

Thus, $f'(x) \ge 0$ in $\left[0, \frac{\pi}{4}\right]$, $f'(x) \le 0$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

and $f'(x) \ge 0$ in $\left[\frac{5\pi}{4}, 2\pi\right]$. **(1)**

Hence, the function is

(i) increasing in $\left[0, \frac{\pi}{4}\right]$ and $\left[\frac{5\pi}{4}, 2\pi\right]$.

(ii) decreasing in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$. **(1)**

14. Find the intervals in which the function given by $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is

(i) increasing.

(ii) decreasing. All Delhi 2012C

Given function is

$$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 4x^3 - 24x^2 + 44x - 24$$
$$= 4(x^3 - 6x^2 + 11x - 6)$$
$$= 4(x - 1)(x^2 - 5x + 6)$$
$$= 4(x - 1)(x - 2)(x - 3) \quad \textbf{(1)}$$

Put $f'(x) = 0$

$$4(x - 1)(x - 2)(x - 3) = 0$$

$\Rightarrow$ $x = 1, 2, 3$

So, the possible intervals are
$(-\infty, 1]$, $[1, 2]$, $[2, 3]$ and $[3, \infty)$. **(1)**

For interval $(-\infty, 1]$, $f'(x) \leq 0$

For interval $[1, 2]$, $f'(x) \geq 0$

For interval $[2, 3]$, $f'(x) \leq 0$

For interval $[3, \infty)$, $f'(x) \geq 0$

(i) Function increases in $[1, 2]$ and $[3, \infty)$. **(1)**

(ii) Function decreases in $(-\infty, 1]$ and $[2\ \ 3]$.
(1)

15. Sand is pouring from the pipe at the rate of $12\ \text{cm}^3/\text{s}$. The falling sand forms a cone on a ground in such a way that the height of cone is always one-sixth of radius of the base. How fast is the height of sand cone increasing when the height is 4 cm? Delhi 2011

Let V be the volume of cone, h be the height and r be the radius of base of the cone.

Given, $\frac{dV}{dt} = 12\text{ cm}^3/\text{s}$ ...(i) **(1/2)**

Also, height of cone = 1/6 (radius of base of cone)

$\therefore \quad h = \frac{1}{6} r$ or $r = 6h$ ...(ii) **(1/2)**

We know that, volume of cone is given by

$$V = \frac{1}{3}\pi r^2 h \quad \text{...(iii)}$$

On putting $r = 6h$ from Eq. (ii) in Eq. (iii), we get

$$V = \frac{1}{3}\pi (6h)^2 \cdot h \Rightarrow V = \frac{\pi}{3} \cdot 36h^3$$

$$\Rightarrow \quad V = 12\pi h^3$$ **(1)**

On differentiating both sides w.r.t. t, we get

$$\frac{dV}{dt} = 12\pi \times 3h^2 \cdot \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dV}{dt} = 36\pi h^2 \cdot \frac{dh}{dt}$$ **(1)**

On putting $\frac{dV}{dt} = 12\text{ cm}^3/\text{s}$ and $h = 4$ cm, we get

$$12 = 36\pi \times 16 \times \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{12}{36\pi \times 16}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{1}{48\pi}\text{ cm/s}$$ **(1)**

Hence, the height of sand cone is increasing at the rate of $1/48\pi$ cm/s.

16. Prove that $y = \frac{4 \sin \theta}{2 + \cos \theta} - \theta$ is an increasing function in $\left[0, \frac{\pi}{2}\right]$. **All India 2011**

> To prove that given function is increasing, prove that $\frac{dy}{d\theta} \geq 0$ for all θ.

Given function is $y = \frac{4 \sin \theta}{2 + \cos \theta} - \theta$ ...(i)

We know that, a function $y = f(x)$ is said to be an increasing function, if $\frac{dy}{dx} \geq 0$, for all values of x.

On differentiating both sides of Eq. (i) w.r.t. θ, we get

$$\frac{dy}{d\theta} = \frac{\left[(2 + \cos \theta) \times \frac{d}{d\theta}(4 \sin \theta) - 4 \sin \theta \times \frac{d}{d\theta}(2 + \cos \theta)\right]}{(2 + \cos \theta)^2} - 1 \quad \textbf{(1/2)}$$

$$= \frac{(2 + \cos \theta)(4 \cos \theta) - 4 \sin \theta (0 - \sin \theta)}{(2 + \cos \theta)^2} - 1$$

$$= \frac{\left[8 \cos \theta + 4 \cos^2 \theta + 4 \sin^2 \theta - (2 + \cos \theta)^2\right]}{(2 + \cos \theta)^2}$$

$$= \frac{\left[8 \cos \theta + 4 (\cos^2 \theta + \sin^2 \theta) - (4 + \cos^2 \theta + 4 \cos \theta)\right]}{(2 + \cos \theta)^2} \quad \textbf{(1)}$$

$[\because (a + b)^2 = a^2 + b^2 + 2ab]$

$$= \frac{8 \cos \theta + 4 - 4 - \cos^2 \theta - 4 \cos \theta}{(2 + \cos \theta)^2}$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$\Rightarrow \quad \frac{dy}{d\theta} = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} \qquad \textbf{(1)}$$

Now, as $\cos\theta \geq 0, \forall\, \theta \in \left[0, \frac{\pi}{2}\right]$.

and $(2+\cos\theta)^2$ being a perfect square is always positive for all $\theta \in \left[0, \frac{\pi}{2}\right]$.

Also, for $\theta \in \left[0, \frac{\pi}{2}\right]$, we know that,

$$0 \leq \cos\theta \leq 1.$$

$$\therefore \quad 4 - \cos\theta > 0 \text{ for all } \theta \in \left[0, \frac{\pi}{2}\right]$$

Hence, we conclude that

$$\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} \geq 0, \forall\, \theta \in \left[0, \frac{\pi}{2}\right].$$

$$\Rightarrow \quad \frac{dy}{d\theta} \geq 0, \forall\, \theta \in \left[0, \frac{\pi}{2}\right].$$

Hence, y is an increasing function in $\left[0, \frac{\pi}{2}\right]$. **(1½)**

17. If the radius of sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area. All India 2011

Let S be the surface area, r be the radius of the sphere.

Given $r = 9$ cm.

Let dr = Approximate error in radius r

and dS = Approximate error in surface area **(1)**

Now, we know that surface area of sphere is given by

$$S = 4\pi r^2$$

On differentiating both sides w.r.t. r, we get

$$\frac{dS}{dr} = 4\pi \times 2r = 8\pi r \quad \textbf{(1)}$$

$$\Rightarrow \quad dS = 8\pi r \cdot dr$$

$$\Rightarrow \quad dS = 8\pi \times 9 \times 0.03$$

$$[\because r = 9 \text{ cm and } dr = 0 \cdot 03 \text{ cm}]$$

$$\Rightarrow \quad dS = 72 \times 0 \cdot 03\pi$$

$$\Rightarrow \quad dS = 2.16\pi$$

Hence, approximate error in surface area is 2.16π cm^2. **(2)**

18. Find the intervals in which the function $f(x) = \sin x + \cos x, 0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

HOTS; Foreign 2011; All India 2009

For strictly decreasing function, $f'(x) < 0$ and for strictly increasing function, $f'(x) > 0$.

The given function is

$$f(x) = \sin x + \cos x, 0 \le x \le 2\pi$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x - \sin x \quad \textbf{(1)}$$

On putting $f'(x) = 0$, we get

$$\cos x - \sin x = 0$$

$$\Rightarrow \quad \cos x = \sin x$$

$$\Rightarrow \quad 1 = \frac{\sin x}{\cos x}$$

$$\Rightarrow \quad 1 = \tan x$$

$$\Rightarrow \quad \tan x = \tan \frac{\pi}{4}$$

$$\text{or} \quad \tan x = \tan \frac{5\pi}{4}$$

$$\left[\because \tan \frac{\pi}{4} = 1 \text{ and } \tan \frac{5\pi}{4} = 1\right]$$

$$\therefore \quad x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \quad \textbf{(1)}$$

Now, we find intervals and check in which interval $f(x)$ is strictly increasing or strictly decreasing.

Interval	**Test value**	**$f'(x) = \cos x - \sin x$**	**Sign of $f'(x)$**
$0 < x < \frac{\pi}{4}$	At $x = \frac{\pi}{6}$	$\cos\frac{\pi}{6} - \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$	+ve
$\frac{\pi}{4} < x < \frac{5\pi}{4}$	At $x = \frac{\pi}{2}$	$\cos\frac{\pi}{2} - \sin\frac{\pi}{2} = 0 - 1 = -1$	–ve
$\frac{5\pi}{4} < x < 2\pi$	At $x = \frac{3\pi}{2}$	$\cos\frac{3\pi}{2} - \sin\frac{3\pi}{2} = 0 - (-1) = 1$	+ve

Since, $f'(x) > 0$ for $0 < x < \frac{\pi}{4}$ and $\frac{5\pi}{4} < x < 2\pi$, so $f(x)$ is strictly increasing in the intervals $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$. While $f'(x) < 0$ in $\frac{\pi}{4} < x < \frac{5\pi}{4}$, so $f(x)$ is strictly decreasing in the interval $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$. **(2)**

NOTE When we check that function f' is increasing/decreasing in an interval, we take any value in that interval and put them in function f' and check whether it is positive/negative.

19. Show that the function $f(x) = x^3 - 3x^2 + 3x$, $x \in R$ is increasing on R. All India 2011C

We know that, a function $y = f(x)$ is said to be increasing on R, if $\frac{dy}{dx} \geq 0, \forall x \in R$. **(1)**

Given, $y = x^3 - 3x^2 + 3x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 6x + 3 \Rightarrow \frac{dy}{dx} = 3(x^2 - 2x + 1)$$

$$\Rightarrow \frac{dy}{dx} = 3(x-1)^2$$ **(1)**

Now, $3(x-1)^2 \geq 0$ for all real values of x, i.e. $\forall x \in R$.

$$\therefore \quad \frac{dy}{dx} \geq 0, \forall x \in R$$

Hence, the given function is increasing on R. **(2)**

20. Find the intervals in which the function $f(x) = (x-1)^3 (x-2)^2$ is

(i) increasing. (ii) decreasing. All India 2011C

Given, $f(x) = (x-1)^3 (x-2)^2$

On differentiating both sides w.r.t. x, we get

$$f'(x) = (x-1)^3 \cdot \frac{d}{dx}(x-2)^2 + (x-2)^2 \cdot \frac{d}{dx}(x-1)^3$$

$$\left[\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$\Rightarrow f'(x) = (x-1)^3 \cdot 2(x-2) + (x-2)^2 \cdot 3(x-1)^2$$

$$= (x-1)^2 (x-2) [2(x-1) + 3(x-2)]$$

$$= (x-1)^2 (x-2) (2x-2+3x-6)$$

$$\Rightarrow f'(x) = (x-1)^2 (x-2)(5x-8)$$

Now, put $f'(x) = 0$

$$\Rightarrow (x-1)^2 (x-2)(5x-8) = 0$$

Either $(x-1)^2 = 0$ or $x-2=0$ or $5x-8=0$

$\therefore \quad x = 1, \frac{8}{5}, 2$ **(1)**

Now, we find intervals and check in which interval $f(x)$ is increasing and decreasing.

Interval	$f'(x) = (x-1)^2 (x-2)(5x-8)$	Sign of $f'(x)$
$x<1$	$(+)(-)(-)$	+ve
$1<x<\frac{8}{5}$	$(+)(-)(-)$	+ve
$\frac{8}{5}<x<2$	$(+)(-)(+)$	–ve
$x>2$	$(+)(+)(+)$	+ve

(1)

We know that, a function $f(x)$ is said to be an increasing function, if $f'(x) \geq 0$ and decreasing, if $f'(x) \leq 0$. So, the given function $f(x)$ is increasing on the intervals $(-\infty, 1]$ $\left[1, \frac{8}{5}\right]$ and

$(2, \infty]$ and decreasing on $\left[\frac{8}{5}, 2\right]$. **(1)**

Since, $f(x)$ is a polynomial function, so it is continuous at $x = 1, \frac{8}{5}, 2$. Hence, $f(x)$ is

(ii) increasing on intervals $(-\infty, 1], \left[1, \frac{8}{5}\right] [2, \infty)$.

(ii) and decreasing on interval $\left[\frac{8}{5}, 2\right]$. **(1)**

21. Find the intervals in which the function $f(x) = 2x^3 + 9x^2 + 12x + 20$ is

(i) increasing. (ii) decreasing. **Delhi 2011C**

The given function is

$$f(x) = 2x^3 + 9x^2 + 12x + 20$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 6x^2 + 18x + 12$$

Put $f'(x) = 0$, we get

$$6x^2 + 18x + 12 = 0$$

$\Rightarrow$ $6(x^2 + 3x + 2) = 0$

$\Rightarrow$ $6(x + 1)(x + 2) = 0$

$\Rightarrow$ $(x + 1)(x + 2) = 0$

$\Rightarrow$ $x + 1 = 0$

or $x + 2 = 0$

$\therefore$ $x = -2, -1$ **(1)**

Now, we find intervals and check in which interval $f(x)$ is increasing and decreasing.

Interval	$f'(x) = 6(x+1)(x+2)$	Sign of $f'(x)$
$x < -2$	$(+)(-)(-)$	+ve
$-2 < x < -1$	$(+)(-)(+)$	−ve
$x > -1$	$(+)(+)(+)$	+ve

(1)

We know that, a function $f(x)$ is said to be an increasing function, if $f'(x) \geq 0$ and decreasing, if $f'(x) \leq 0$. So, given function is increasing on intervals $(-\infty, -2]$ and $[-1, \infty)$ and decreasing on interval $[-2, -1]$. **(1)**

Since, $f(x)$ is a polynomial function, so it is continuous at $x = -1, -2$.

Hence, given function is

(i) increasing on intervals $(-\infty, -2]$ and $[-1, \infty)$.

(ii) decreasing on interval $[-2, -1]$. **(1)**

22. Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x - 15$ is

(i) increasing. (ii) decreasing. **Delhi 2011C**

The given function is

$$f(x) = 2x^3 - 9x^2 + 12x - 15$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 6x^2 - 18x + 12$$

On putting $f'(x) = 0$, we get

$$6x^2 - 18x + 12 = 0$$

$\Rightarrow$ $6(x^2 - 3x + 2) = 0$

$\Rightarrow$ $6(x - 1)(x - 2) = 0$

$\Rightarrow$ $x - 1 = 0$ or $x - 2 = 0$

$\therefore$ $x = 1, 2$ **(1)**

Now, we find intervals and check in which intervals $f(x)$ is increasing or decreasing.

Interval	$f'(x) = 6(x-1)(x-2)$	Sign of $f'(x)$
$x<1$	$(+)(-)(-)$	+ve
$1<x<2$	$(+)(+)(-)$	−ve
$x>2$	$(+)(+)(+)$	+ve

(1)

Now, we know that, a function $f(x)$ is increasing when $f'(x) \geq 0$ and it is decreasing when $f'(x) \leq 0$. So, the given function is increasing on intervals $(-\infty, 1)$ and $(2, \infty)$ and decreasing on intervals $(1, 2)$. **(1)**

Since, $f(x)$ is a polynomial function, so it is continuous at $x = 1, 2$.

Hence, given function is

(i) increasing on intervals $(-\infty, 1]$ and $[2, \infty)$.

(ii) decreasing on interval $[1, 2]$. **(1)**

23. Find the intervals in which the function $f(x) = 2x^3 - 15x^2 + 36x + 17$ is increasing or decreasing. All India 2010C

Do same as Que. 22.

[**Ans.** Increasing on $(-\infty, 2]$ and $[3, \infty)$ and decreasing on $[2,3]$.]

24. Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x + 15$ is

(i) increasing. (ii) decreasing.

All India 2010C, 2008, 2008C

Do same as Que. 22.

[**Ans.** (i) Increasing on $(-\infty, 1]$ and $[2, \infty)$

(ii) Decreasing on $[1,2]$.]

25. Find the intervals in which the function $f(x) = (x-1)(x-2)^2$ is increasing or decreasing. All India 2009C

The given function is $f(x) = (x-1)(x-2)^2$

On differentiating both sides w.r.t. x, we get

$$f'(x) = (x-1) \times \frac{d}{dx}(x-2)^2 + (x-2)^2 \times \frac{d}{dx}(x-1) \quad \textbf{(1)}$$

$$\Rightarrow \quad f'(x) = (x-1)\,2\,(x-2) + (x-2)^2 \cdot 1$$

$$\Rightarrow \quad f'(x) = 2(x-1)(x-2) + (x-2)^2$$

$$= (x-2)[2x - 2 + x - 2]$$

$$\Rightarrow \quad f'(x) = (x-2)(3x-4)$$

On putting $f'(x) = 0$, we get

$$(x-2)(3x-4) = 0$$

$$\Rightarrow \quad x - 2 = 0$$

or $$3x - 4 = 0$$

$$\therefore \quad x = \frac{4}{3} \text{ or } 2 \quad \textbf{(1)}$$

Now, we find the intervals in which $f(x)$ is increasing or decreasing.

Interval	$f'(x)=(x-2)(3x-4)$	Sign of $f'(x)$
$x<\frac{4}{3}$	(–) (–)	+ve
$\frac{4}{3}<x<2$	(–) (+)	–ve
$x>2$	(+) (+)	+ve

(1)

We know that, a function $f(x)$ is said to be an increasing function, if $f'(x)\geq 0$ and a decreasing function when $f'(x)\leq 0$. So, $f(x)$ is increasing on $\left(-\infty, \frac{4}{3}\right)$ and $(2, \infty)$ and decreasing on $\left(\frac{4}{3}, 2\right)$.

Since, $f(x)$ is a polynomial function, so it is continuous at $x = 4/3$ and 2.

Hence, given function is increasing on intervals $\left(-\infty, \frac{4}{3}\right]$ and $[2, \infty)$ and decreasing on interval $\left[\frac{4}{3}, 2\right]$. **(1)**

26. Find the intervals in which $f(x)=x^3-12x^2+36x+17$ is increasing or decreasing function. **Delhi 2009C**

Do same as Que. 23.

[**Ans.** Increasing on $(-\infty,2]$ and $[6,\infty)$ and decreasing on $[2,6]$.]

27. The length x of a rectangle is decreasing at the rate of 5 cm/min and the width y is increasing at the rate of 4 cm/min. When $x = 8$ cm and $y = 6$ cm, find the rate of change of

(i) the perimeter. (ii) area of rectangle.

HOTS; All India 2009C

> ? Using the relation perimeter of rectangle, $P = 2(x + y)$ and area of rectangle, $A = xy$, differentiate both sides with respect to t and put them in rate of change value and get the result.

Given that length x of a rectangle is decreasing at the rate of 5 cm/min.

$$\therefore \quad \frac{dx}{dt} = -5 \text{ cm/min} \quad \text{...(i)}$$

Also, the breadth y of rectangle is increasing at the rate of 4 cm/min.

$$\therefore \quad \frac{dy}{dt} = 4 \text{ cm/min} \quad \text{...(ii)} \quad \textbf{(1/2)}$$

(i) Here, we have to find rate of change of perimeter, i.e. dP/dt **(1)**

and we know that, perimeter $P = 2(x + y)$

On differentiating both sides w.r.t. t, we get

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

28. Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}, x \neq 0$ is

(i) increasing.

(ii) decreasing. Delhi 2009C

Given function is $f(x) = x^3 + \frac{1}{x^3}, x \neq 0$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 3x^2 - \frac{3}{x^4} \quad \textbf{(1)}$$

On putting $f'(x) = 0$, we get

$$3x^2 - \frac{3}{x^4} = 0 \Rightarrow \frac{3x^6 - 3}{x^4} = 0$$

$$\Rightarrow 3x^6 - 3 = 0 \Rightarrow 3x^6 = 3$$

$$\Rightarrow x^6 = 1 \Rightarrow x = \pm 1$$

$$\begin{bmatrix} \because x^6 = (1)^6 & \text{and} & x^6 = (-1)^6 \\ \Rightarrow x = 1 & \text{and} & x = -1 \end{bmatrix} \textbf{(1)}$$

Now, we find intervals in which $f(x)$ is increasing or decreasing.

Interval	$f'(x) = \frac{3x^6 - 3}{x^4}$	Sign of $f'(x)$
$x < -1$	$\frac{(+)}{(+)}$	+ve
$-1 < x < 1$, $x \neq 0$	$\frac{(-)}{(+)}$	−ve
$x > 1$	$\frac{(+)}{(+)}$	+ve

Now, we know that a function $f(x)$ is increasing when $f'(x) \geq 0$ and it is said to be decreasing when $f'(x) \leq 0$. So, $f(x)$ is increasing on intervals $(-\infty, -1)$ and $(1, \infty)$ and it is decreasing on $(-1, 1) - \{0\}$.

Also, $f(x)$ is continuous at $x = 1, -1$.

Hence, $f(x)$ is

(i) increasing on intervals $(-\infty, -1]$ and $[1, \infty)$.

(ii) decreasing on interval $[-1, 1] - \{0\}$. **(1)**

29. If $f(x) = 3x^2 + 15x + 5$, then find the approximate value of $f(3.02)$ using differentials. Delhi 2008C

Do same as Que. 9. **[Ans.** 77.66]

6 Marks Questions

30. Prove that the function f defined by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in $(-1, 1)$. Hence, find the intervals in which $f(x)$ is

(i) strictly increasing. (ii) strictly decreasing.

Delhi 2014C

Given function is $f(x) = x^2 - x + 1$ **(1)**

On differentiating w.r.t. x, we get

$$f'(x) = 2x - 1$$ **(1)**

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$ **(1)**

$\Rightarrow$ $2x - 1 > 0 \Rightarrow x < 1/2$

So, $f(x)$ is decreasing on $(-\infty, 1/2]$. **(1)**

Hence, $f(x)$ is neither increasing nor decreasing in $[-1, 1]$. **(2)**

31. Find the intervals in which the function $f(x) = 20 - 9x + 6x^2 - x^3$ is

(i) strictly increasing. (ii) strictly decreasing.

All India 2010

The given function is $f(x) = 20 - 9x + 6x^2 - x^3$.

On differentiating both sides w.r.t. x, we get

$$f'(x) = -9 + 12x - 3x^2 \quad \textbf{(1)}$$

On putting $f'(x) = 0$, we get

$$-9 + 12x - 3x^2 = 0$$

$$\Rightarrow \quad -3(x^2 - 4x + 3) = 0$$

$$\Rightarrow \quad -3(x-1)(x-3) = 0$$

$$\Rightarrow \quad (x-1)(x-3) = 0$$

$$\Rightarrow \quad x - 1 = 0 \text{ or } x - 3 = 0$$

$$\therefore \quad x = 1 \text{ or } 3 \quad \textbf{(2)}$$

Now, we find intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$f'(x) = -3(x-1)(x-3)$	**Sign of** $f'(x)$
$x < 1$	$(-)(-)(-)$	$-$ve
$1 < x < 3$	$(-)(+)(-)$	$+$ve
$x > 3$	$(-)(+)(+)$	$-$ve

(1)

Now, we know that, a function $f(x)$ is said to be strictly increasing when $f'(x) > 0$ and it is said to be strictly decreasing, if $f'(x) < 0$. So, the given function $f(x)$ is

(i) strictly increasing on the interval $(1, 3)$ and

(ii) strictly decreasing on the intervals $(-\infty, 1)$ and $(3, \infty)$. **(2)**

32. Find the intervals in which the function f given by $f(x) = \sin x - \cos x, 0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

Delhi 2010

Given function is

$$f(x) = \sin x - \cos x, 0 \le x \le 2\pi$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x + \sin x \quad \textbf{(1)}$$

On putting $f'(x) = 0$, we get

$$\cos x + \sin x = 0 \Rightarrow \sin x = -\cos x$$

$$\Rightarrow \frac{\sin x}{\cos x} = -1 \Rightarrow \tan x = -1$$

For $x \in [0, 2\pi]$, $\tan x = \tan \frac{3\pi}{4} \Rightarrow x = \frac{3\pi}{4}$

or $\tan x = \tan \frac{7\pi}{4} \Rightarrow x = \frac{7\pi}{4}$

$$\therefore \quad x = \frac{3\pi}{4}, \frac{7\pi}{4} \quad \textbf{(2)}$$

[$\because \tan\theta$ is -1 in 2nd quadrant and 4th quadrant]

Now, we find the intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	Test value	$f'(x) = \cos x + \sin x$	Sign of $f'(x)$
$0 < x < \frac{3\pi}{4}$	At $x = \frac{\pi}{2}$	$f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + \sin\frac{\pi}{2}$ $= 0 + 1 = 1$	+ve
$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	At $x = \frac{5\pi}{6}$	$f'\left(\frac{5\pi}{6}\right) = \cos\frac{5\pi}{6} + \sin\frac{5\pi}{6}$ $= \cos\left(\pi - \frac{\pi}{6}\right) + \sin\left(\pi - \frac{\pi}{6}\right)$ $= -\cos\frac{\pi}{6} + \sin\frac{\pi}{6}$ $= \frac{-\sqrt{3}}{2} + \frac{1}{2} = \frac{-\sqrt{3}+1}{2}$	−ve
$\frac{7\pi}{4} < x < 2\pi$	At $x = 2\pi$	$f'(2\pi) = \cos 2\pi + \sin 2\pi$ $= \cos(2\pi - 0°) + \sin(2\pi - 0°)$ $= \cos 0° - \sin 0° = 1 - 0 = 1$	+ve

(2)

We know that, a function $f(x)$ is said to be strictly increasing in an interval when $f'(x) > 0$ and it is said to be strictly decreasing when $f'(x) < 0$. So, the given function $f(x)$ is strictly increasing in intervals $\left(0, \frac{3\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$ and it is strictly decreasing in the interval $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$. **(1)**